Converse elimination in the algebra of binary relations

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Calculus of relations

Natural set of (primitive) operations on binary relations (graphs) [Peirce, Schröder, Tarski] over some domain V

$$\begin{aligned}
id &= \{(x, x) \mid x \in V\} \\
r \cup s &= \{(x, y) \mid (x, y) \in r \vee (x, y) \in s\} \\
r \circ s &= \{(x, y) \mid \exists z : (x, z) \in r \wedge (z, y) \in s\} \\
r^c &= \{(x, y) \in V^2 \mid (x, y) \notin r\} \\
r^{-1} &= \{(x, y) \mid (y, x) \in r\} \\
r^+ &= \text{the transitive closure of } r
\end{aligned}$$

Derived operators

$$
\begin{aligned}
di &= \{(x,y) \in V^2 \mid x \neq y\} = id^c \\
all &= V^2 = id \cup di \\
r \cap s &= \{(x,y) \mid (x,y) \in r \wedge (x,y) \in s\} = (r^c \cup s^c)^c \\
r - s &= \{(x,y) \mid (x,y) \in r \wedge (x,y) \notin s\} = r \cap s^c \\
\pi_1(r) &= \{(x,x) \mid \exists y : (x,y) \in r\} = (r \circ all) \cap id \\
\pi_2(r) &= \{(x,x) \mid \exists y : (y,x) \in r\} = (all \circ r) \cap id
\end{aligned}
$$

Fragments

A fragment is a set of primitive or derived operations

The most basic fragment $\mathcal{N}$ is $\{id, \cup, \circ\}$

$\mathcal{N}(F)$ denotes $\mathcal{N}$ extended with the operations in F

We say that F has an operator if the operator is in F or if it can be constructed from other operators in F

- E.g. $r \cap s = r - (r - s)$
- E.g. $\pi_1(r) = (r \circ r^{-1}) \cap id$

Queries

Fix a binary relational vocabulary Γ (Label set)

Expressions over a fragment F are built from relation names in Γ using the operations in F

$\Rightarrow$ Map instances of Γ (graphs) to binary relations

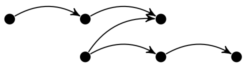
Well established logic: $\mathcal{N}(^{-1}, ^{c}) \equiv \mathrm{FO}^3$ (Tarski & Givant)

Boolean queries

Boolean queries (graph properties): test nonemptiness of expression results

Examples:

- Transitivity: $all - (all \circ ((R \circ R) - R) \circ all) \neq \emptyset$
- S-T connectivity: $S \circ R^{+} \circ T \neq \emptyset$
- $R^2 \circ R^{-1} \circ R^2 \neq \emptyset$ matches the pattern



Converse elimination

Which boolean queries expressed using $^{-1}$ can be equivalently expressed without using $^{-1}$?

E.g. $R^2 \circ R^{-1} \circ R^2 \neq \emptyset$ is equivalent to $\pi_1(R^2 \circ \pi_2(\pi_1(R^2) \circ R)) \neq \emptyset$

Formally: a fragment F admits converse elimination if every boolean query in $\mathcal{N}(F)$ can be expressed in $\mathcal{N}(F - \{^{-1}\})$

Theorem ([Fletcher et al.])

If F has projection, but neither intersection nor transitive closure, then F admits converse elimination

Note: the theorem clearly applies to the example above

Motivation

- Comparing different computational models/logics as is done in complexity theory and finite model theory
- E.g. π allows looking back (π_2) but not navigation back. Is looking back always sufficient to eliminate converse?

Remaining questions

Theorem ([Fletcher et al.])

If F has projection, but neither intersection nor transitive closure, then F admits converse elimination

1. What if intersection is present?
2. What if transitive closure is present?
3. Expression complexity of converse elimination?

Remaining questions

Theorem ([Fletcher et al.])

If F has projection, but neither intersection nor transitive closure, then F admits converse elimination

1. What if intersection is present?

   ⇒ Converse elimination fails: $(R^2 \circ R^{-1} \circ R^2) \cap R \neq \emptyset$ is not expressible in the most powerful fragment without converse $(\mathcal{N}(^c, ^+))$

2. What if transitive closure is present?

   ⇒ We will prove that it also fails

3. Expression complexity of converse elimination?

   ⇒ We will prove an exponential blowup in degree

Converse elimination fails in the presence of transitive closure

Theorem

The query that checks for the existence of the graph pattern

is not expressible in the largest language without converse.
Formally, $R^2 \circ (R \circ R^{-1})^+ \circ R^2 \neq \emptyset$ *is not expressible in* $\mathcal{N}({}^c, {}^+)$

Bisimulation game for $\mathcal{N}(^c)$

Restricted version of the 3-pebble game for FO^3

Intuition: relax invariant condition of partial isomorphism so that converse relations need not be preserved by Duplicator

Let $\mathrm{G}_1 = (G_1, a_1, b_1)$ and $\mathrm{G}_2 = (G_2, a_2, b_2)$ be pointed graphs. A k-round bisimulation game on G_1 and G_2 proceeds as follows:

- The spoiler picks an $i \in \{1, 2\}$ and a node x_i in G_i
- The duplicator responds with a node x_j ($j \neq i$) in G_j
- Continue two $k-1$-round subgames on:
  - (G_1, a_1, x_1) and (G_2, a_2, x_2)
  - (G_1, x_1, b_1) and (G_2, x_2, b_2)

k-bisimilarity and indistinguishability of pointed graphs

The duplicator wins the game if for any newly started subgame on (G_1, y_1, z_1) and (G_2, y_2, z_2) we have

- $(y_1, z_1) \in R(G_1)$ iff $(y_2, z_2) \in R(G_2)$ for every $R \in \Gamma$
- $y_1 = z_1$ iff $y_2 = z_2$

If the duplicator has a winning strategy for the k-round bisimulation game on G_1 and G_2, we say that G_1 and G_2 are k-bisimilar

Theorem ([Fletcher et al.])

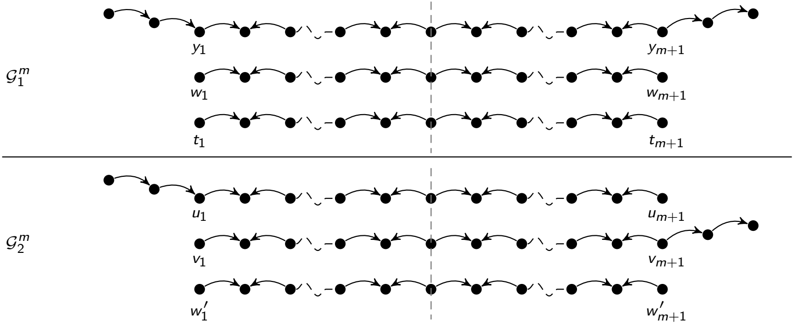
(G_1, a_1, b_1) and (G_2, a_2, b_2) are k-bisimilar iff for any expression $e \in \mathcal{N}(^c)$ with degree at most k:

$$(a_1, b_1) \in e(G_1) \Leftrightarrow (a_2, b_2) \in e(G_2)$$

* degree(e) is the maximum number of nested compositions

Family of graphs to prove inexpressibility

Let $Q : R^2 \circ (R \circ R^{-1})^+ \circ R^2$ and define the family $\mathcal{G}_1^m$ and $\mathcal{G}_2^m$:



$\Rightarrow$ $Q(\mathcal{G}_1^m) \neq \emptyset$ and $Q(\mathcal{G}_2^m) = \emptyset$

Bisimilarity result on $\mathcal{G}_1^m$ and $\mathcal{G}_2^m$

Theorem

For any $a_1, b_1 \in \mathcal{G}_1^m$ there exists $a_2, b_2 \in \mathcal{G}_2^m$ such that $(\mathcal{G}_1, a_1, b_1)$ and $(\mathcal{G}_2^m, a_2, b_2)$ are $m/2 - 1$-bisimilar

- Main technical contribution. Proof is long and technical

Corollary

A boolean query in $\mathcal{N}(^c)$ with degree at most $m/2 - 1$ cannot be true on $\mathcal{G}_1^m$ and false on $\mathcal{G}_2^m$ simultaneously

Proof that converse elimination fails in presence of TC

Corollary

A boolean query in $\mathcal{N}(^c)$ with degree at most $m/2 - 1$ cannot be true on G_1^m and false on G_2^m simultaneously

Recall: $Q = R^2 \circ (R \circ R^{-1})^+ \circ R^2$

Suppose $e \neq \emptyset$ with $e \in \mathcal{N}(^c, ^+)$ expresses $Q \neq \emptyset$. For any m define $k_m = |\mathcal{G}_1^m| = |\mathcal{G}_2^m|$.

- For any n there exists e_n without $^+$ equivalent to e on graphs G with domain size at most n. ($e_{k_m} \neq \emptyset$ is equivalent to $Q \neq \emptyset$ on $\mathcal{G}_1^m$ and $\mathcal{G}_2^m$)
  - $\Rightarrow$ Can ensure that degree(e_n) is logarithmic in n
- $\Rightarrow$ exists l such that degree(e_{k_l}) $\leq l/2 - 1$
  - $\Rightarrow$ By the Corollary e_{k_l} is nonempty on both $\mathcal{G}_1^l$ and $\mathcal{G}_2^l$
- $\Rightarrow$ Contradicts that $e_{k_l} \neq \emptyset$ is equivalent to $Q \neq \emptyset$ on $\mathcal{G}_1^l$ and $\mathcal{G}_2^l$

Exponential blowup in degree for converse elimination

Consider the family $Q_n : R^2 \circ (R \circ R^{-1})^n \circ R^2 \neq \emptyset$

- Degree of Q_n is $O(\log(n))$
- Converse elimination for this family cannot be done in degree $o(n)$

Corollary

Converse elimination in the language with projection, but with neither intersection nor transitive closure is exponential in the degree

Directions for further research

1. We know that converse elimination is exponential in the degree. Can we establish a similar result in the length?
2. Another interesting derived operator: Residuations [Pratt. Origins of calculus of binary relations]
   C/B is the maximal X such that
   $$X \circ B \subseteq C$$
   $B \backslash C$ is the maximal X such that
   $$B \circ X \subseteq C$$
   - How does $\mathcal{N}(/, \backslash)$ compare to $\mathcal{N}(^c)$?
   - Satisfiability and equivalence problem in $\mathcal{N}(/, \backslash)$?

References

This presentation is part of a larger project on the calculus of relations:

G.H.L. Fletcher, M. Gyssens, D. Leinders, D. Surinx, J. Van den Bussche, D. Van Gucht, S. Vansummeren, and Y. Wu.
Relative expressive power of navigational querying on graphs.
Information Sciences, 298:390–406, 2015.

G.H.L. Fletcher, M. Gyssens, D. Leinders, J. Van den Bussche, D. Van Gucht, and S. Vansummeren.
Similarity and bisimilarity notions appropriate for characterizing indistinguishability in fragments of the calculus of relations.
Journal of Logic and Computation, 25(3):549–580, 2015.

G.H.L. Fletcher, M. Gyssens, D. Leinders, J. Van den Bussche, D. Van Gucht, S. Vansummeren, and Y. Wu.
The impact of transitive closure on the expressiveness of navigational query languages on unlabeled graphs.
Annals of Mathematics and Artificial Intelligence, 73(1–2):167–203, 2015.

D. Surinx, G.H.L. Fletcher, M. Gyssens, D. Leinders, J. Van den Bussche, D. Van Gucht, S. Vansummeren, and Y. Wu.
Relative expressive power of navigational query on graphs using transitive closure.
Logic Journal of the IGPL, 23(5):759–788, 2015.