Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis

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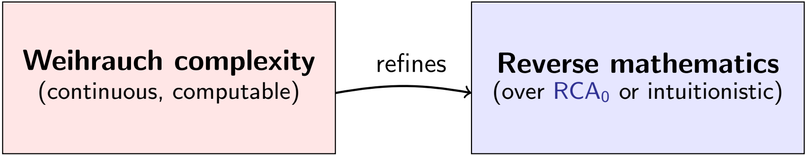
Context of this Seminar



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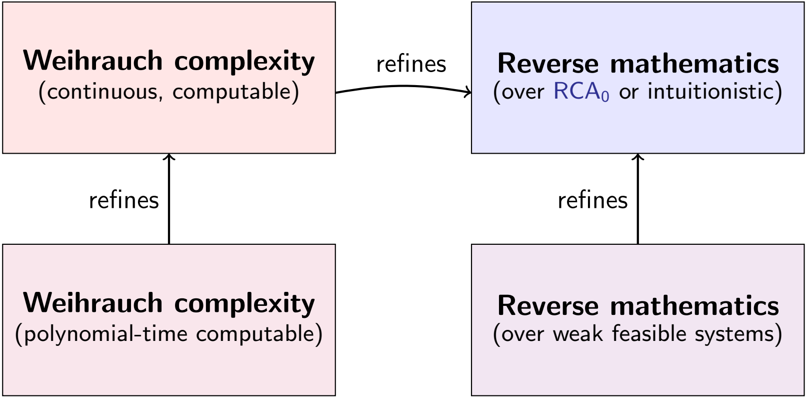

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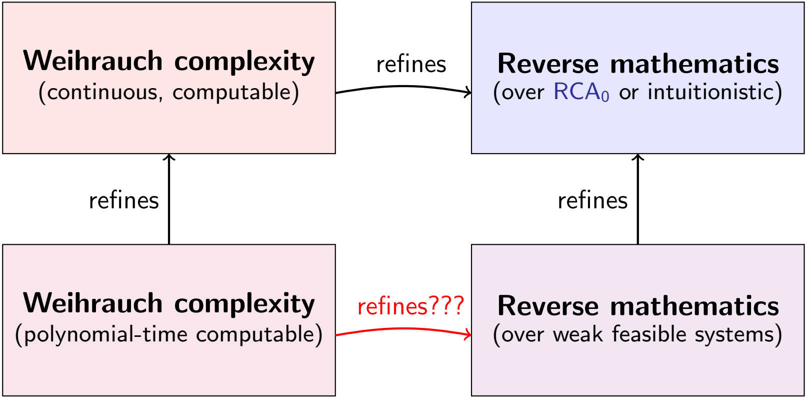
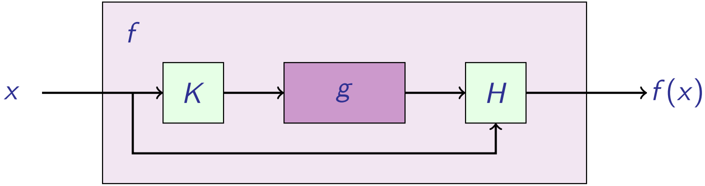


Open Problem

Does polynomial-time Weihrauch reducibility lead to a refinement of reverse mathematics over a suitable weak feasible system?

Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $H(x, gK(x)) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(H(\mathrm{id}, gK))$.
- f is strongly Weihrauch reducible to g, $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $HgK(x) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(HgK)$.
- Analogously, one can define Weihrauch reducibility with continuous H, K or polynomial-time computable H, K or other categories of maps.

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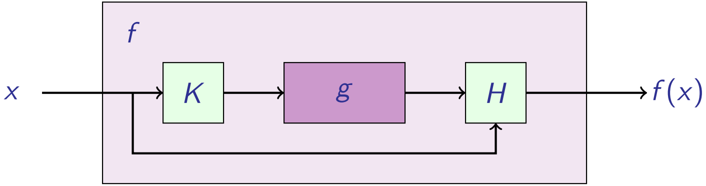

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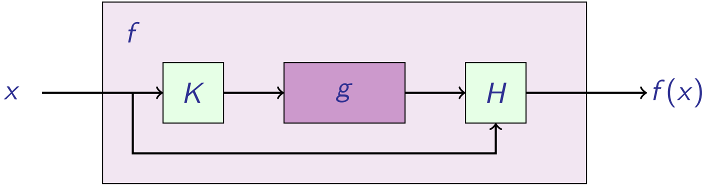

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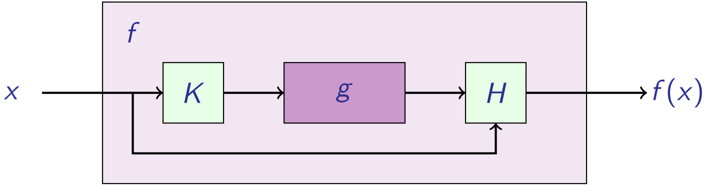

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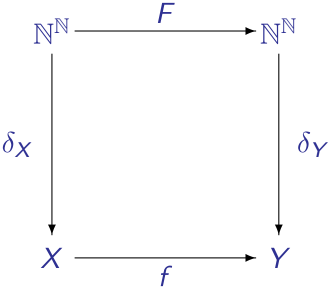
Weihrauch Reducibility

- A representation of X is a surjective map $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$.
- $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f :\subseteq X \rightrightarrows Y$, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.



- f is continuous, computable, polynomial-time computable or Borel measurable, if there it admits a corresponding realizer F.
- $f \leq_W g \iff$ there are computable $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $H\langle \mathrm{id}, GK\rangle \vdash f$ whenever $G \vdash g$.
- $f \leq_W g \iff \{F : F \vdash f\} \leq_W \{G : G \vdash g\}$.

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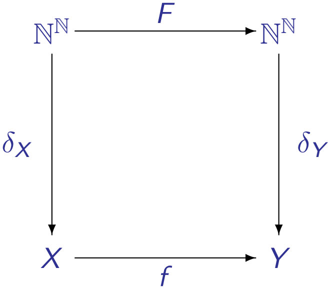

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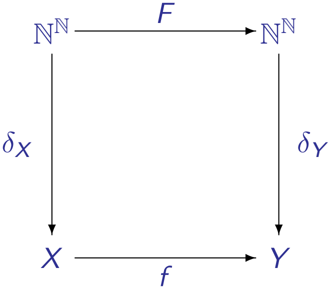

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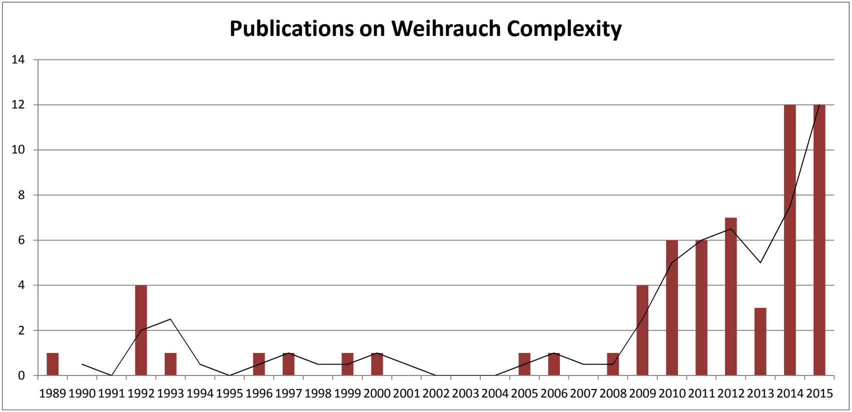
- $f \leq_\mathrm{W} g \iff \{F : F \vdash f\} \leq_\mathrm{W} \{G : G \vdash g\}$.

Some History on Weihrauch Reducibility

- 1992 Klaus Weihrauch introduced the concept of his reducibility for single-valued functions $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and for sets of such functions (in two unpublished technical reports).
- 1989-2007 he supervised 6 MSc/PhD theses on this topic, mostly unpublished (von Stein, Mylatz, B., Hertling, Pauly).
- The reducibility was also considered for single-valued functions $f :\subseteq X \to Y$ on other topological/represented spaces.
- 2008 Guido Gherardi and Alberto Marcone noticed that this reducibility for multi-valued functions can be used to classify the computational content of Π_2 theorems.
- 2009 Akitoshi Kawamura (and Stephen Cook) rediscovered a polynomial-time version of Weihrauch reducibility and used it for the study of uniform computational time complexity.
- 2012 Dorais, Dzhafarov, Hirst, Mileti, Shafer rediscovered Weihrauch reducibility directly for the special case of Π^1_2 statements (work extended by Hirschfeldt and Jockusch).

Publications on Weihrauch Complexity



Based on the bibliography `http://cca-net.de/publications/weibib.php`, currently there are 64 entries, please help to complete this!

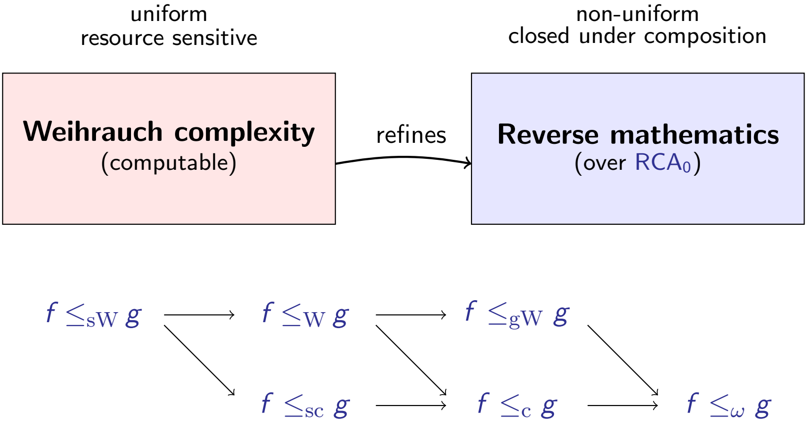
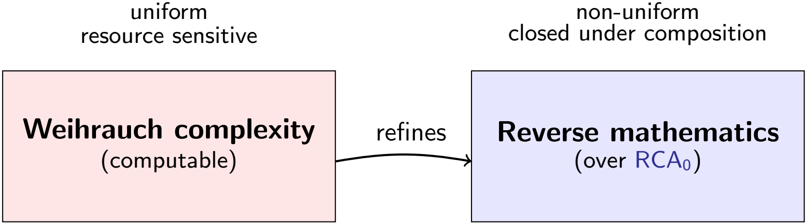


Diagram based on: Hirschfeldt and Jockusch, On Notions of Computability Theoretic Reduction Between Π^1_2 Principles, preprint 2015.

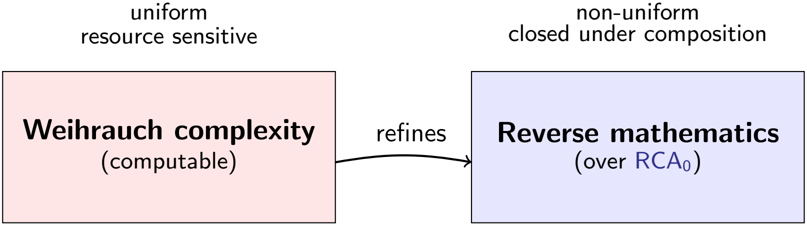
Logical Interpretation



Open Problem

Can the slogan "Weihrauch complexity is a kind of a model of reverse mathematics with some form of (intuitionistic) linear logic" be converted into a theorem?

Logical Interpretation



The Medvedev lattice is a model of some intermediate logic.

Proposition (B. and Gherardi 2009)

$A \leq_{\mathrm{M}} B \iff c_A \leq_{\mathrm{W}} c_B \iff \mathrm{id}|_B \leq_{\mathrm{W}} \mathrm{id}|_A.$

Algebraic Operations in the Weihrauch Lattice

Definition

Let f, g be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of f and g, only one of the solutions will be provided (Sum)
- $f * g$: f and g can be used consecutively (Comp. Product)
- $g \to f$: this is the simplest problem h such that f can be reduced to $g * h$ (Implication)
- f^*: f can be used any given finite number of times in parallel (Star)
- $\widehat{f}$: f can be used countably many times in parallel (Parallelization)
- f': f can be used on the limit of the input (Jump)

Some Formal Definitions

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z, (x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z, z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z, (x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*, f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum (B., Gherardi, Pauly).
- Parallelization $\widehat{\ }$ and star operation * are closure operators in the Weihrauch lattice (B., Gherardi, Pauly).
- With $\sqcup, \times, ^*$ one obtains a Kleene algebra (B., Pauly).
- The Weihrauch lattice is neither a Brouwer nor a Heyting algebra (Higuchi und Pauly 2012).

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Open Problem

Does the strong Weihrauch reducibility induce a lattice structure?

- It is known that $\sqcap$ is an infimum for $\leq_{\mathrm{sW}}$ and hence one obtains a lower semi-lattice (B., Gherardi).
- One can show that $\sqcup$ fails as supremum for $\leq_{\mathrm{sW}}$.

Closure Operators and Reducibilities

Remark

There is a vague analogy between versions of Weihrauch reducibilities induced by closure operators and computability theoretic reducibilities:

Closure operation	**Reducibility**
$f \leq_{\mathrm{sW}} g$	*one-one reducibility*
$f \leq_{\mathrm{W}} g$	*many-one reducibility*
$f \leq_{\mathrm{W}} g^*$	*weak truth-table reducibility*
$f \leq_{\mathrm{W}} \widehat{g}$	*Turing reducibility*
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Can this analogy be made more precise?

Note: we already have an embedding of Turing degrees into parallelizable Weihrauch degrees.

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Infima and Suprema

The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

Definition (Compositional product, implication)

For two mathematical problem f, g we define the

- $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f$ and $g_0 \leq_{\mathrm{W}} g\}$ and
- $g \to f := \min\{h : f \leq_{\mathrm{W}} g * h\}$.

The maximum and minimum is understood with respect to $\leq_{\mathrm{W}}$ and they always exist (B. and Pauly 2013).

Examples of Mathematical Problems

- The Limit Problem is the mathematical problem

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, ...\rangle \mapsto \lim_{i\to\infty} p_i$$

  with $\mathrm{dom}(\lim) := \{(x_i) : (x_i) \text{ is convergent}\}$.

- Martin-Löf Randomness is the mathematical problem $\mathrm{MLR} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathrm{MLR}(x) := \{y \in 2^{\mathbb{N}} : y \text{ is Martin-Löf random relative to } x\}.$$

- The Cohesiveness Problem is the mathematical problem $\mathrm{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ where $\mathrm{COH}(R_i)$ contains all infinite $X \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ one of the sets

$$X \cap R_i \text{ or } X \cap (\mathbb{N} \setminus R_i)$$

  is finite.

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Theorems as Problems

Definition

Any theorem T of the Π_2 form

$$(\forall x \in X)(\exists y \in Y)\ (x \in D \Longrightarrow P(x, y))$$

is identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and

$$F(x) := \{y \in Y : P(x, y)\}.$$

Example: Weak Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WWKL} :\subseteq \mathsf{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathsf{Tr} : \mu([T]) > 0\}$.

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Study some basic theorems that involve more quantifiers!

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Choice

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The choice problem $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X$ of a topological space X is the mathematical problem induced by the statement:

- Every non-empty closed set $A \subseteq X$ has a point $x \in A$.

The choice problem is related to the zero problem of finding a solution $x \in X$ of the equation

$$f(x) = 0$$

for a continuous function $f : X \to \mathbb{R}$. Formally, we consider the zero problem as $\mathsf{Z}_X : \mathcal{C}(X) \rightrightarrows X, f \mapsto f^{-1}\{0\}$.

Proposition

$\mathsf{Z}_X \equiv_{\mathrm{sW}} \mathsf{C}_X$ *for every computable metric space* X.

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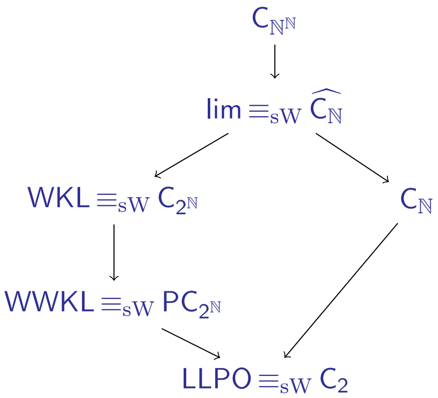
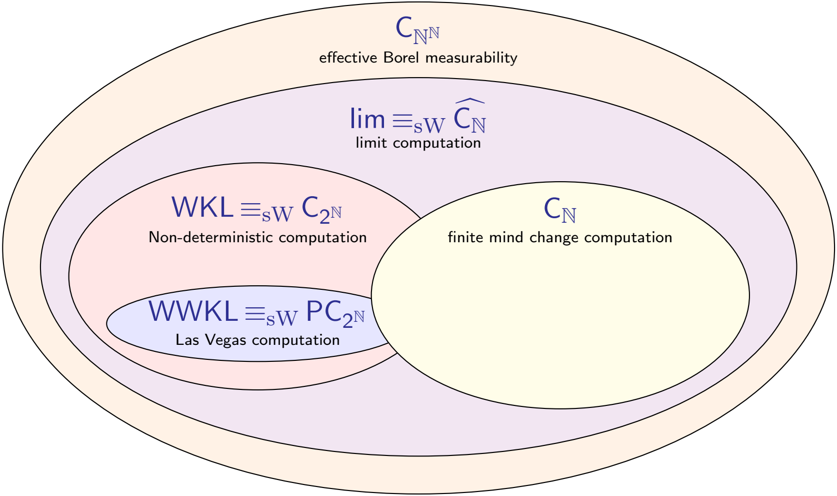
Computational Classes



Computational Classes



Calibrating Computability with Choice

Theorem (B., de Brecht and Pauly 2010)

Let f be a mathematical problem. Then:

- $f \leq \mathsf{C}_{\{0\}} \iff f$ *is computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *comp. with finitely many mind changes,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \iff f$ *is non-deterministically computable,*
- $f \leq_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \iff f$ *is limit computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \iff f$ *is effectively Borel measurable.*

The last mentioned items holds for single-valued f on Polish spaces.

Corollary (B., Gherardi, Hölzl 2015)

$f \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} \iff f$ *is Las Vegas computable.*

The Weihrauch Lattice refines the Borel Hierarchy

Theorem (B. 2005)

- $f \leq_{\mathrm{W}} \lim^{(n)} \iff f$ *is effectively* $\mathbf{\Sigma}^0_{n+2}$*–measurable.*

reducibility	**hierarchy**
many-one	arithmetical
Weihrauch	effective Borel

The Weihrauch Lattice refines the Borel Hierarchy

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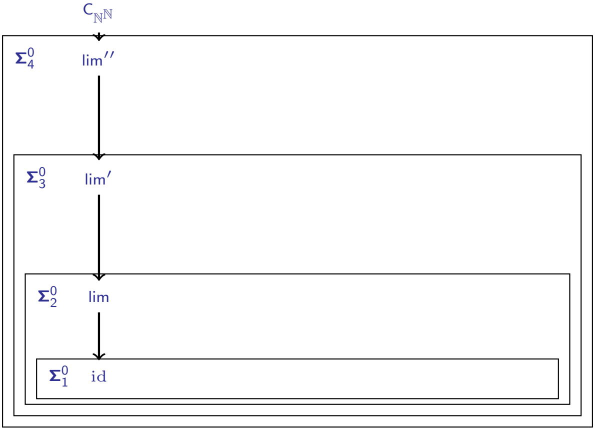

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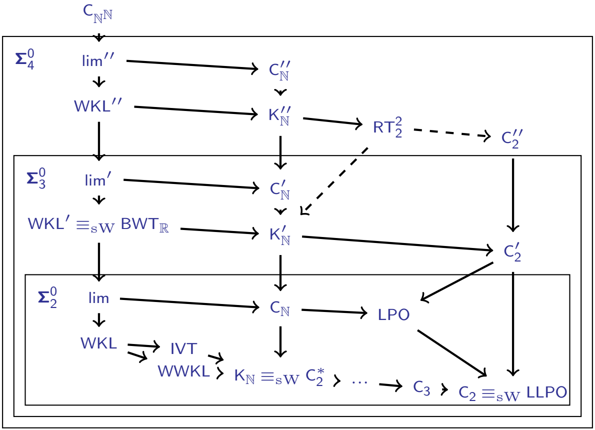
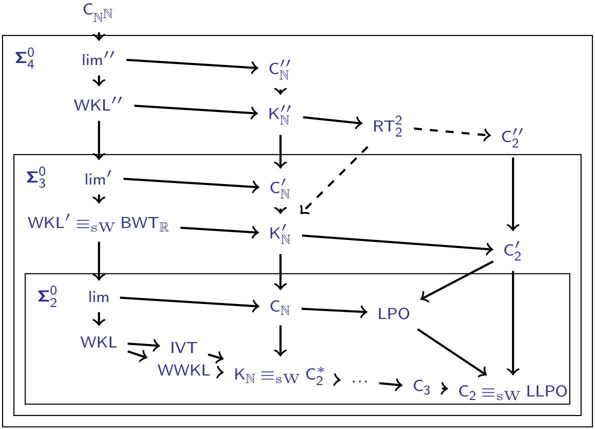


Open Problem

Does $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{RT}^2_2$ *hold? (We have* $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{SRT}^2_\mathbb{N}$*.)*

The Weihrauch Lattice refines the Borel Hierarchy



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Weihrauch Complexity Versus Reverse Mathematics

Weihrauch compl.	**Reverse maths.**	**Intuitionistic maths.**
$\mathsf{C}_1 \equiv_W \mathrm{id}$	RCA_0	constructive
$\mathsf{C}_{2^{\mathbb{N}}}$	WKL_0	LLPO
$\mathsf{K}_{\mathbb{N}}^{(n)}$	$\mathsf{B}\Sigma^0_{n+1}$	
$\mathsf{C}_{\mathbb{N}}^{(n)}$	$\mathsf{I}\Sigma^0_{n+1}$	
$\lim^{(n)}$, $n \in \mathbb{N}$	ACA_0	LPO
$\bigsqcup_{n=0}^{\infty} \lim^{(n)}$	ACA_0'	
$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$	ATR_0?	
	closed under $*$	closed under $*, \widehat{\ }$

Open Problem

Does $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ correspond to ATR_0 or rather to some other higher system?

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A Technique

Choice Elimination as a Separation Tool

Theorem (B., de Brecht and Pauly 2012)

$f \leq_W C_{2^{\mathbb{N}}} \times g \Longrightarrow f \leq_W g$

for single-valued $f :\subseteq X \to Y$ *on metric spaces* X, Y.

Corollary

$C_{\mathbb{N}} \not\leq_W C_{2^{\mathbb{N}}}$.

Theorem (Le Roux and Pauly 2013)

$f \leq_W C_{\mathbb{N}} * g \Longrightarrow f \leq_W g$

for total fractals f.

Corollary

$C_{2^{\mathbb{N}}} \,|_W\, C_{\mathbb{N}}$.

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$\mathsf{C}_{2^{\mathbb{N}}} \mid_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

Classification of Theorems

Choice on Natural Numbers

Theorem (B. and Gherardi 2012)

The following problems and theorems are Weihrauch equivalent:

- *The choice problem* $C_{\mathbb{N}}$ *on natural numbers.*
- *The Baire Category Theorem* BCT.
- *The Banach Inverse Mapping Theorem* IMT.
- *The Open Mapping, Closed Graph and Uniform Boundedness Theorems.*

All for computable normed spaces.

The entire equivalence class shares the following features:

- All members map computable inputs to (some) computable outputs.
- All members are not uniformly computable.
- All members are computable with finitely many mind changes.
- All members have parallelizations that are equivalent to the limit map and they are closed under composition.

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Choice on Cantor Space

Theorem

The following problems and theorems are Weihrauch equivalent:

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- *Weak Kőnig's Lemma* WKL.
- *The Hahn-Banach Theorem* HBT
  (Gherardi and Marcone 2009)
- *The Brouwer-Fixed Point Theorem* BFT *for dimension* ≥ 2.
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The entire equivalence class shares the following features:

- All members map computable inputs to (some) low outputs.
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- *The differentiability problem* $d :\subseteq \mathcal{C}[0,1] \to \mathcal{C}[0,1], f \mapsto f'$ *(von Stein 1989).*
- *The Monotone Convergence Theorem* MCT.
- *The Fréchet-Riesz Theorem for Hilbert spaces. (follows from B. and Yoshikawa 2006)*
- *The Radon-Nikodym Theorem. (Hoyrup, Rojas and Weihrauch 2012)*

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Contraposition Matters

The Heine-Borel Theorem can be formalized in different version, which are contrapositive to each other:

- $\mathsf{HB}_0 :\subseteq \mathcal{O}([0,1])^{\mathbb{N}} \rightrightarrows \mathbb{N}, (U_i)_i \mapsto \{n \in \mathbb{N} : [0,1] \subseteq \bigcup_{i=0}^{n} U_i\}$, where $\mathrm{dom}(\mathsf{HB}_0)$ contains all sequences of open subsets $(U_i)_i$ that cover $[0,1]$.
- $\mathsf{HB}_1 :\subseteq \mathcal{O}([0,1])^{\mathbb{N}} \rightrightarrows [0,1], (U_i)_i \mapsto [0,1] \setminus \bigcup_{i=0}^{\infty} U_i$, where $\mathrm{dom}(\mathsf{HB}_1)$ contains all sequences of open subsets $(U_i)_i$ that do not contain a finite subcover of $[0,1]$.

Proposition

$\mathsf{HB}_0 \equiv_{\mathrm{W}} \mathrm{id}$ *is computable* $\mathsf{HB}_1 \equiv_{\mathrm{W}} \mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}}$.

Similar situations have been studied in detail for

- The Baire Category Theorem (B., Hendtlass, Kreuzer)
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Some Examples on Algebraic Relations

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- $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WKL}'$ *(B., Gherardi and Marcone 2012)*
- $\mathsf{AoUC}^*_{[0,1]} \equiv_{\mathrm{W}} \mathsf{RDIV}^* \equiv_{\mathrm{W}} \mathsf{NASH}$ *(Pauly 2010)*
- $\widehat{\mathsf{RT}^2_2} \equiv_{\mathrm{W}} \mathsf{WKL}''$ *and* $\widehat{\mathsf{RT}^n_2} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$ *(Rakotoniaina 2015)*
- $\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$ *(B. and Pauly 2013)*
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Find further problems that can be characterized in a natural way with the help of implication $\to$.

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Polynomial-Time Complexity

Polynomial-Time Results

Theorem (Stephen A. Cook und Akitoshi Kawamura 2012)

- max *complete in* $\mathsf{FP}^{\mathsf{NP}}$ *with respect to polynomial-time Weihrauch reducibility.*
- $\int$ *is complete in* $\mathsf{FP}^{\#\mathsf{P}}$ *with respect to polynomial-time Weihrauch reducibility.*

- $\max : C[0,1]^2 \to C[0,1], f \mapsto \max(f)$ is defined by $\max(f) : [0,1] \to \mathbb{R}, x \mapsto \max_{y \in [0,1]} f(x,y)$.
- $\int : C[0,1]^2 \to C[0,1], f \mapsto \int f$ is defined by $\int f : [0,1] \to \mathbb{R}, x \mapsto \int_0^1 f(x,y)dy$.

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Characterize the computational time complexity of computable theorems in analysis and compare the results to reverse mathematics over weak systems.

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Bibliography styles: alphabetically, chronologically, by publication type. BibTeX file: wei.bib

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